Multisolitons for coupled Lowest Landau Level equations

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Introduction

Motivation : Exhibit **unbounded trajectories** of the NLS-like system

$$\begin{cases} i\partial_t u - Hu = \mathcal{N}(u), \quad (t,z) \in \mathbb{R} \times \mathbb{R}^2, \\ u(0,\cdot) = u_0, \end{cases}$$

where $z = x + iy$ and

$$H = -(\partial_x^2 + \partial_y^2) + (x^2 + y^2),$$

is the harmonic oscillator.

- Joint work with Valentin Schwinte
- Previous work with Patrick Gérard and Pierre Germain

Introduction

Notations : consider the 2D **harmonic oscillator**

$$H = -(\partial_x^2 + \partial_y^2) + (x^2 + y^2) = -4\partial_z \partial_{\bar{z}} + |z|^2,$$

where $z = x + iy$, $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$. This operator acts on the space

$$\left\{ u(z) = e^{-\frac{|z|^2}{2}} f(z) \,,\ f \text{ entire holomorphic} \right\} \cap \mathscr{S}'(\mathbb{C}).$$

The **Bargmann-Fock space** $\mathcal{E}$ is

$$\mathcal{E} = \left\{ u(z) = e^{-\frac{|z|^2}{2}} f(z) \,,\ f \text{ entire holomorphic} \right\} \cap L^2(\mathbb{C}).$$

The **special Hermite functions** $(\varphi_n)_{n \geq 0}$ are

$$\varphi_n(z) = \frac{z^n}{\sqrt{\pi n!}} e^{-\frac{|z|^2}{2}}.$$

They form a **Hilbertian basis** of $\mathcal{E}$ and

$$H\varphi_n = 2(n+1)\varphi_n, \quad n \geq 0.$$

For $s \geq 0$, we define the **harmonic Sobolev spaces** by

$$\mathcal{H}^s = \{u \in \mathscr{S}'(\mathbb{C}),\ H^{s/2}u \in L^2(\mathbb{C})\} \cap \mathcal{E}.$$

We have, with $\langle z \rangle = (1+|z|^2)^{1/2}$

$$\mathcal{H}^s = L^{2,s}_{\mathcal{E}} := \{u \in \mathscr{S}'(\mathbb{C}),\ \langle z \rangle^s u \in L^2(\mathbb{C})\} \cap \mathcal{E},$$

with the **equivalence of norms**

$$c\|\langle z \rangle^s u\|_{L^2(\mathbb{C})} \leq \|u\|_{\mathcal{H}^s} \leq C\|\langle z \rangle^s u\|_{L^2(\mathbb{C})}, \quad \forall\, u \in L^{2,s}_{\mathcal{E}},$$

Hypercontractivity estimates (Carlen) : for all $1 \leq p \leq q \leq +\infty$

$$\Big(\frac{q}{2\pi}\Big)^{1/q} \|u\|_{L^q(\mathbb{C})} \leq \Big(\frac{p}{2\pi}\Big)^{1/p} \|u\|_{L^p(\mathbb{C})}.$$

Introduction

Denote by Π the **orthogonal projector** on the space $\mathcal{E}$

$$(\Pi u)(z) = \frac{1}{\pi} e^{-\frac{|z|^2}{2}} \int_{\mathbb{C}} e^{\overline{w}z - \frac{|w|^2}{2}} u(w)\, dL(w),$$

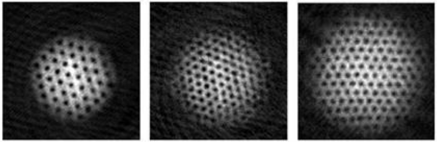
and consider the coupled **Lowest Landau Level** equations

$$\begin{cases} i\partial_t u = \Pi(|v|^2 u), & (t,z) \in \mathbb{R} \times \mathbb{C}, \\ i\partial_t v = \sigma \Pi(|u|^2 v), \\ u(0,\cdot) = u_0 \in \mathcal{E},\ v(0,\cdot) = v_0 \in \mathcal{E}. \end{cases} \tag{1}$$

Hamiltonian :

$$\mathcal{H}(u,v) = \int_{\mathbb{C}} |u|^2 |v|^2 dL.$$

This system is used in the modeling of **fast rotating Bose-Einstein condensates**.



Symmetries / conservation laws

- **Phase rotations**

$$T_{\theta_1,\theta_2} : (u,v)(z) \mapsto \big(e^{i\theta_1}u(z), e^{i\theta_2}v(z)\big) \qquad \text{for } (\theta_1,\theta_2) \in \mathbb{T}^2$$

Mass

$$M(u) = \int_{\mathbb{C}} |u(z)|^2 dL(z), \quad M(v) = \int_{\mathbb{C}} |v(z)|^2 dL(z).$$

- **Space rotations**

$$L_\theta : (u,v)(z) \mapsto \big(u(e^{i\theta}z), v(e^{i\theta}z)\big) \qquad \text{for } \theta \in \mathbb{T}$$

Angular momentum

$$P_\sigma(u,v) = \int_{\mathbb{C}} \big(|z|^2 - 1\big)\big(|u(z)|^2 + \sigma|v(z)|^2\big) dL(z).$$

Symmetries / conservation laws

- **Magnetic translations**

$$R_\alpha : (u,v)(z) \mapsto \left(u(z+\alpha)e^{\frac{1}{2}(\bar{z}\alpha - z\bar{\alpha})}, v(z+\alpha)e^{\frac{1}{2}(\bar{z}\alpha - z\bar{\alpha})}\right) \qquad \text{for } \alpha \in \mathbb{C}$$

Magnetic momentum

$$Q_\sigma(u,v) = \int_{\mathbb{C}} z\left(|u(z)|^2 + \sigma|v(z)|^2\right) dL(z).$$

Symmetries / conservation laws

- **Magnetic translations**

$$R_\alpha : (u,v)(z) \mapsto \big(u(z+\alpha)e^{\frac{1}{2}(\bar{z}\alpha - z\bar{\alpha})}, v(z+\alpha)e^{\frac{1}{2}(\bar{z}\alpha - z\bar{\alpha})}\big) \qquad \text{for } \alpha \in \mathbb{C}$$

Magnetic momentum

$$Q_\sigma(u,v) = \int_{\mathbb{C}} z\big(|u(z)|^2 + \sigma|v(z)|^2\big)\, dL(z).$$

- If $u(z) = f(z)e^{-|z|^2/2} \in \mathcal{E}$, then

$$R_\alpha u(z) = f(z+\alpha)e^{-z\overline{\alpha} - |\alpha|^2/2 - |z|^2/2} \in \mathcal{E}$$

- For all $p \geq 1$,

$$\|R_\alpha u\|_{L^p} = \|u\|_{L^p}$$

- **But** for all $s > 0$

$$\|\langle z\rangle^s R_\alpha u\|_{L^2} = \|\langle z-\alpha\rangle^s u\|_{L^2} \sim \alpha^s \|u\|_{L^2}, \quad \alpha \longrightarrow +\infty.$$

Global well-posedness result

Theorem

For every $(u_0, v_0) \in L^{2,1}_{\mathcal{E}} \times L^{2,1}_{\mathcal{E}}$, *there exists a unique solution* $(u, v) \in \mathcal{C}^\infty(\mathbb{R}, L^{2,1}_{\mathcal{E}} \times L^{2,1}_{\mathcal{E}})$ *to the system* (1).

Moreover :

$$M(u) = \int_{\mathbb{C}} |u(t,z)|^2 dL(z) = M(u_0)$$

$$M(v) = \int_{\mathbb{C}} |v(t,z)|^2 dL(z) = M(v_0)$$

$$\mathcal{H}(u,v) = \int_{\mathbb{C}} |u(t,z)|^2 |v(t,z)|^2 dL(z) = \mathcal{H}(u_0, v_0)$$

$$P_\sigma(u,v) = \int_{\mathbb{C}} (|z|^2 - 1)\left(|u(t,z)|^2 + \sigma |v(t,z)|^2\right) dL(z) = P_\sigma(u_0, v_0)$$

$$Q_\sigma(u,v) = \int_{\mathbb{C}} z\left(|u(t,z)|^2 + \sigma |v(t,z)|^2\right) dL(z) = Q_\sigma(u_0, v_0).$$

Global well-posedness result : idea of the proof

We consider the equation

$$\begin{cases} i\partial_t u = \Pi(|u|^2 u), & (t,z) \in \mathbb{R} \times \mathbb{C}, \\ u(0,\cdot) = u_0. \end{cases}$$

We find a fixed point of the mapping

$$F : u \longmapsto u_0 - i \int_0^t \Pi(|u|^2 u)(s)\, ds.$$

Key ingredient : Carlen estimate : $\|u\|_{L^6} \leq C \|u\|_{L^2}$

$$\begin{aligned} \|F(u)(t)\|_{L^2} &\leq \|u_0\|_{L^2} + \int_0^t \|\Pi(|u|^2 u)(s)\|_{L^2} ds \\ &\leq \|u_0\|_{L^2} + \int_0^t \left\|(|u|^2 u)(s)\right\|_{L^2} ds \\ &= \|u_0\|_{L^2} + \int_0^t \|u\|_{L^6}^3 ds \\ &\leq \|u_0\|_{L^2} + C \int_0^t \|u\|_{L^2}^3 ds. \end{aligned}$$

Proof of the Carlen estimate $\|u\|_{L^6(\mathbb{C})} \leq C\|u\|_{L^2(\mathbb{C})}$

For $u \in \mathcal{E}$ we have

$$u(z) = \Pi u(z) = \frac{1}{\pi}\int_{\mathbb{C}} e^{-\frac{|z|^2}{2}+\overline{w}z-\frac{|w|^2}{2}} u(w)\, dL(w),$$

and since $|e^{-\frac{|z|^2}{2}+\overline{w}z-\frac{|w|^2}{2}}| = e^{-\frac{|z-w|^2}{2}}$, we get

$$|u(z)| \leq \frac{1}{\pi}\int_{\mathbb{C}} e^{-\frac{|z-w|^2}{2}} |u(w)|\, dL(w) = \big(\psi \star |u|\big)(z),$$

where $\psi(z) = \frac{1}{\pi} e^{-|z|^2/2} \in L^r(\mathbb{C})$ for any $r \in [1, +\infty)$.

We can conclude with the Young inequality

$$\|u\|_{L^6(\mathbb{C})} \leq \big\|\psi \star |u|\big\|_{L^6(\mathbb{C})} \leq \|\psi\|_{L^{3/2}(\mathbb{C})}\|u\|_{L^2(\mathbb{C})} \leq C\|u\|_{L^2(\mathbb{C})}.$$

Bounds on the Sobolev norms : defocusing case

$$\begin{cases} i\partial_t u = \Pi(|v|^2 u), & (t,z) \in \mathbb{R} \times \mathbb{C}, \\ i\partial_t v = \Pi(|u|^2 v), \\ u(0,\cdot) = u_0 \in \mathcal{E}, \ v(0,\cdot) = v_0 \in \mathcal{E}. \end{cases} \tag{2}$$

Theorem (Schwinte-Thomann)

Assume that $\sigma = 1$. Let $k \geq 0$ be an integer and $(u_0, v_0) \in L^{2,k}_{\mathcal{E}} \times L^{2,k}_{\mathcal{E}}$. Then there exists a unique solution $(u,v) \in \mathcal{C}^\infty(\mathbb{R}, L^{2,k}_{\mathcal{E}} \times L^{2,k}_{\mathcal{E}})$ to equation (2) and it satisfies for all $t \in \mathbb{R}$,

$$\|\langle z\rangle^k u(t)\|_{L^2(\mathbb{C})} + \|\langle z\rangle^k v(t)\|_{L^2(\mathbb{C})} \lesssim (1+|t|)^{\frac{k-1}{4}} \qquad \text{if} \quad k \geq 3$$

$$\|\langle z\rangle^2 u(t)\|_{L^2(\mathbb{C})} + \|\langle z\rangle^2 v(t)\|_{L^2(\mathbb{C})} \lesssim (1+|t|)^{\frac{1}{2}} \qquad \text{if} \quad k = 2.$$

Bounds on the Sobolev norms : idea of the proof

Show that ($k \geq 2$)

$$\frac{d}{dt}\left(\|\langle z\rangle^k u\|^2_{L^2(\mathbb{C})} + \|\langle z\rangle^k v\|^2_{L^2(\mathbb{C})}\right) \lesssim \left(\|\langle z\rangle^k u\|^2_{L^2(\mathbb{C})} + \|\langle z\rangle^k v\|^2_{L^2(\mathbb{C})}\right)^{1-\frac{2}{k-1}}$$

+ integration.

- Key ingredient 1)

$$\big\|\langle z\rangle u\big\|_{L^2(\mathbb{C})} + \big\|\langle z\rangle v\big\|_{L^2(\mathbb{C})} \lesssim \big\|\langle z\rangle u_0\big\|_{L^2(\mathbb{C})} + \big\|\langle z\rangle v_0\big\|_{L^2(\mathbb{C})}$$

- Key ingredient 2)

$$\|u\|_{L^\infty(\mathbb{C})} \lesssim \|u\|_{L^2(\mathbb{C})}.$$

- Key ingredient 3)

$$\big\|(-\Delta)^k(|u|^2)\big\|_{L^\infty(\mathbb{C})} \lesssim \|u\|^2_{L^\infty(\mathbb{C})}$$

Bounds on the Sobolev norms : focusing case

$$\begin{cases} i\partial_t u = \Pi(|v|^2 u), \quad (t,z) \in \mathbb{R} \times \mathbb{C}, \\ i\partial_t v = -\Pi(|u|^2 v), \\ u(0,\cdot) = u_0 \in \mathcal{E},\ v(0,\cdot) = v_0 \in \mathcal{E}. \end{cases} \tag{3}$$

Theorem (Schwinte-Thomann)

Assume that $\sigma = -1$. Let $k \geq 0$ be an integer and $(u_0, v_0) \in L^{2,k}_{\mathcal{E}} \times L^{2,k}_{\mathcal{E}}$. Then there exists a unique solution $(u, v) \in \mathcal{C}^\infty(\mathbb{R}, L^{2,k}_{\mathcal{E}} \times L^{2,k}_{\mathcal{E}})$ to equation (3) and it satisfies for all $t \in \mathbb{R}$,

$$\big\|\langle z\rangle^k u(t)\big\|_{L^2(\mathbb{C})} \leq \big\|\langle z\rangle^k u_0\big\|_{L^2(\mathbb{C})}\big(1 + C\|v_0\|^2_{L^2(\mathbb{C})}|t|\big)^k$$
$$\big\|\langle z\rangle^k v(t)\big\|_{L^2(\mathbb{C})} \leq \big\|\langle z\rangle^k v_0\big\|_{L^2(\mathbb{C})}\big(1 + C\|u_0\|^2_{L^2(\mathbb{C})}|t|\big)^k.$$

- Same proof but we can **no more** use the conservation of the norm

$$\big\|\langle z\rangle u\big\|_{L^2(\mathbb{C})} + \big\|\langle z\rangle v\big\|_{L^2(\mathbb{C})}$$

Existence of progressive waves

Growth of Sobolev norms ?

We look for progressive waves solution to (1)

$$\big(u(t,z), v(t,z)\big) = \big(e^{-i\lambda t}U(z+\alpha t)e^{\frac{1}{2}(\bar{z}\alpha - z\bar{\alpha})t}, e^{-i\mu t}V(z+\alpha t)e^{\frac{1}{2}(\bar{z}\alpha - z\bar{\alpha})t}\big).$$

Theorem (Schwinte-Thomann)

Assume that $\sigma = -1$. Then

$$\begin{cases} U = K\big(\frac{1}{2} + \frac{\sqrt{3}}{2} iz\big)e^{-|z|^2/2} \\ V = K\big(\frac{1}{2} - \frac{\sqrt{3}}{2} iz\big)e^{-|z|^2/2}, \end{cases}$$

defines a progressive wave with speed $\alpha = \frac{\sqrt{3}}{32\pi}K^2 \neq 0$.

- Direct computions
- Uniqueness with a finite number of zeros, up to symmetries.

Existence of progressive waves : motivation

Let $\sigma \in \{1, -1\}$.

Assume that

$$\big(u(t,z), v(t,z)\big) = \big(e^{-i\lambda t} U(z+\alpha t) e^{\frac{1}{2}(\bar{z}\alpha - z\overline{\alpha})t}, e^{-i\mu t} V(z+\alpha t) e^{\frac{1}{2}(\bar{z}\alpha - z\overline{\alpha})t}\big).$$

Then

$$Q_\sigma(U,V) = Q_\sigma(R_{\alpha t}U, R_{\alpha t}V) = Q_\sigma(U,V) - \alpha t\big(M(U) + \sigma M(V)\big)\,,$$

and

$$P_\sigma(U,V) = P_\sigma(R_{\alpha t}U, R_{\alpha t}V) =$$
$$= P_\sigma(U,V) - t\big(\alpha\overline{Q_\sigma(U,V)} + \overline{\alpha}Q_\sigma(U,V)\big) + t^2|\alpha|^2\big(M(U) + \sigma M(V)\big),$$

- This implies $\sigma = -1$
- This also implies : $M(U) = M(V)$ and $\mathfrak{Re}\big(\overline{\alpha}Q_-(U,V)\big) = 0$.

Consequence : growth of Sobolev norms

$$\begin{cases} i\partial_t u = \Pi(|v|^2 u), \quad (t,z) \in \mathbb{R}\times\mathbb{C}, \\ i\partial_t v = -\Pi(|u|^2 v), \\ u(0,\cdot) = u_0 \in \mathcal{E},\ v(0,\cdot) = v_0 \in \mathcal{E}. \end{cases} \tag{3}$$

Corollary

Assume that $\sigma = -1$. Let (U, V) be previously defined, then corresponding solution (u, v) to (3) satisfies for all $s \geq 0$

$$\|\langle z\rangle^s u(t)\|_{L^2(\mathbb{C})} \sim c_s |t|^s, \quad \|\langle z\rangle^s v(t)\|_{L^2(\mathbb{C})} \sim c_s |t|^s, \quad t \longrightarrow \pm\infty.$$

- For all $s > 0$

$$\|\langle z\rangle^s R_{\alpha t} U\|_{L^2} = \|\langle z - \alpha t\rangle^s U\|_{L^2} \sim \alpha^s |t|^s \|u\|_{L^2}, \quad \alpha \longrightarrow +\infty.$$

Consequence : growth of Sobolev norms

$$\begin{cases} i\partial_t u = \Pi(|v|^2 u), \quad (t,z) \in \mathbb{R}\times\mathbb{C}, \\ i\partial_t v = -\Pi(|u|^2 v), \\ u(0,\cdot) = u_0 \in \mathcal{E},\ v(0,\cdot) = v_0 \in \mathcal{E}. \end{cases} \tag{3}$$

Corollary

Assume that $\sigma = -1$. Let (U, V) be previously defined, then corresponding solution (u, v) to (3) satisfies for all $s \geq 0$

$$\|\langle z\rangle^s u(t)\|_{L^2(\mathbb{C})} \sim c_s|t|^s, \quad \|\langle z\rangle^s v(t)\|_{L^2(\mathbb{C})} \sim c_s|t|^s, \quad t \longrightarrow \pm\infty.$$

- Similar results for the system

$$\begin{cases} i\partial_t u - Hu = \Pi(|v|^2 u), \quad (t,z) \in \mathbb{R}\times\mathbb{C}, \\ i\partial_t v - Hv = \sigma\Pi(|u|^2 v), \\ u(0,z) = u_0(z),\ v(0,z) = v_0(z). \end{cases}$$

Well posedness for exponentially localised functions

Proposition

Let $\kappa \geq 0$. Assume that $(u_0, v_0) \in \mathcal{X}_{\mathcal{E}}^{\kappa} \times \mathcal{X}_{\mathcal{E}}^{\kappa}$, then the corresponding solution to (3) satisfies $(u, v) \in \mathcal{C}^{\infty}\big(\mathbb{R}, \mathcal{X}_{\mathcal{E}}^{\kappa} \times \mathcal{X}_{\mathcal{E}}^{\kappa}\big)$. Moreover, for every $t \in \mathbb{R}$,

$$\|e^{\kappa|z|}u(t)\|_{L^2(\mathbb{C})} \leq \|e^{\kappa|z|}u_0\|_{L^2(\mathbb{C})} e^{c_\kappa \|v_0\|_{L^2}^2 |t|}$$

$$\|e^{\kappa|z|}v(t)\|_{L^2(\mathbb{C})} \leq \|e^{\kappa|z|}v_0\|_{L^2(\mathbb{C})} e^{c_\kappa \|u_0\|_{L^2}^2 |t|} \ ,$$

where the constant $c_\kappa > 0$ only depends on $\kappa > 0$ (notice that $c_0 = 0$);

Existence of multi-solitons

Theorem

Let $n \geq 1$. For $1 \leq j \leq n$, let $\alpha_j \in \mathbb{C}^*$. Assume that $\alpha_j \neq \alpha_\ell$ for $j \neq \ell$. Denote by

$$\alpha_\sharp = \min_{j \neq \ell} |\alpha_j - \alpha_\ell|.$$

Then, for all $\kappa > 0$, there exists a solution $(u, v) \in \mathcal{C}^\infty\big(\mathbb{R}, \mathcal{X}_\mathcal{E}^\kappa \times \mathcal{X}_\mathcal{E}^\kappa\big)$ to equation (3) of the form

$$\begin{cases} u(t,z) = \displaystyle\sum_{j=1}^{n} e^{-i\lambda_j t} U_j(z + \alpha_j t) e^{\frac{1}{2}(\bar{z}\alpha_j - z\overline{\alpha_j})t} + r_1(t,z) \\ v(t,z) = \displaystyle\sum_{j=1}^{n} e^{-i\mu_j t} V_j(z + \alpha_j t) e^{\frac{1}{2}(\bar{z}\alpha_j - z\overline{\alpha_j})t} + r_2(t,z), \end{cases}$$

where for all $c < \frac{1}{4}$ and all $m \in \mathbb{N}$, there exists $C_{m,\kappa} > 0$ such that

$$\left\| e^{\kappa|z|}(\partial_t^m r_1)(t) \right\|_{L^2} + \left\| e^{\kappa|z|}(\partial_t^m r_2)(t) \right\|_{L^2} \leq C_{m,\kappa} e^{-c\alpha_\sharp^2 t^2}, \quad t \geq 0.$$

Existence of multi-solitons

- Inspired from the work of Ferriere 2020 on NLS with logarithmic nonlinearity : existence of **Gaussons**.
- Error analysis and backward time integration learnt from Faou-Raphaël 2020.
- What about $t \longrightarrow -\infty$?
- We have uniqueness result if $\kappa \gg 1$.

Application to the linear harmonic oscillator

The previous result allows us to give new examples of unbounded trajectories to the **2D linear harmonic oscillator**

$$\begin{cases} i\partial_t \psi - H\psi + V(t,x,y)\psi = 0, & (t,x,y) \in \mathbb{R} \times \mathbb{R}^2, \\ \psi(0,\cdot) = \psi_0 \in L^2(\mathbb{R}^2). \end{cases} \tag{4}$$

Theorem

Assume that $\alpha_j \neq \alpha_\ell$ *for* $j \neq \ell$. *Then there exists a potential* $V \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^2; \mathbb{R})$ *such that for all* $\sigma \geq 0$ *and all* $k \in \mathbb{N}$

$$\lim_{t \to +\infty} \|\partial_t^k V(t)\|_{\mathcal{H}^\sigma(\mathbb{C})} = 0,$$

and there exists a solution $\psi \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^2; \mathbb{C})$ *to the equation* (4) *of the form*

$$\psi(t) = \sum_{j=1}^{n} e^{-i\lambda_j \ln t} e^{-2it} L_{-2t} R_{\alpha_j \ln t} U_j + \eta(t),$$

where $\|\eta(t)\|_{\mathcal{H}^1(\mathbb{C})} \longrightarrow 0$, *when* $t \longrightarrow +\infty$.

Theorem

Assume that $\alpha_j \neq \alpha_\ell$ *for* $j \neq \ell$. *Then there exists a potential* $V \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^2; \mathbb{R})$ *such that for all* $\sigma \geq 0$ *and all* $k \in \mathbb{N}$

$$\lim_{t \to +\infty} \|\partial_t^k V(t)\|_{\mathcal{H}^\sigma(\mathbb{C})} = 0,$$

and there exists a solution $\psi \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^2; \mathbb{C})$ *to the equation* (4) *of the form*

$$\psi(t) = \sum_{j=1}^{n} e^{-i\lambda_j \ln t} e^{-2it} L_{-2t} R_{\alpha_j \ln t} U_j + \eta(t),$$

where $\|\eta(t)\|_{\mathcal{H}^1(\mathbb{C})} \longrightarrow 0$, *when* $t \longrightarrow +\infty$.

- The function ψ is a sum of space-localised bubbles and

$$\|\psi(t)\|_{\mathcal{H}^1(\mathbb{C})} \sim \Big(\sum_{j=1}^{n} c_j\Big) \ln t, \quad c_j > 0, \quad t \longrightarrow +\infty.$$

- This is a direct application of a result of Faou-Raphaël 2020.

Some references

Study of the LLL/CR equation

- Aftalion-Blanc-Nier 2006, Nier 2007 $\longrightarrow$ LLL
- Faou-Germain-Hani 2016 $\longrightarrow$ Derivation of CR
- Germain-Hani-Thomann 2016 $\longrightarrow$ Link between CR and LLL
- Biasi-Bizon-Craps-Evnin 2017-2019 $\longrightarrow$ LLL
- Gérard-Germain-Thomann 2019 $\longrightarrow$ LLL
- Schwinte-Thomann 2020 $\longrightarrow$ Coupled LLL

Growth of Sobolev norms for NLS-like equations

- Resonant NLS : Colliander-Keel-Staffilani-Takaoka-Tao 2010
- NLS : Hani 2014, Hani-Pausader-Tzvetkov-Visciglia 2015, Carles-Gallagher 2018
- Szegö : Gérard-Grellier 2014, Xu 2014, Xu 2017, Thirouin 2019.
- Thomann 2020
- Faou-Raphaël 2020

Some references

Multi-solitons for NLS

- Côte, Le Coz, Martel, Merle, ... Survey by Martel.
- Ferriere 2020 (log NLS)

Some references

Growth of norms for linear Schrödinger operators

- Bourgain 1999 (Schrödinger on the torus)
- Delort 2014
- Maspero-Robert 2017 (upper bounds)
- Bambusi-Grébert-Maspero-Robert 2018
- Maspero 2019
- Haus-Maspero 2020
- Liang-Zhao-Zhou 2020
- Faou-Raphaël 2020